FLUID FLOW IN THE ENTRANCE REGION OF A CIRCULAR TUBE

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Inzhenerno-Fizicheskii Zhurnal, Vol. 15, No. 1, pp. 98–102, 1968

UDC 532.542

Newtonian fluid flow in the entrance region of a circular tube is investigated.

A model of the entrance region of a Newtonian fluid flow is shown in Fig. 1.

In the inlet section of a tube of radius R let the fluid velocity be everywhere constant and equal to U_0. We assume that the velocity U in the flow core is a function of the longitudinal coordinate z only, and that in the boundary layer the velocity profile varies according to the law

$$v = A(z)[1 - \xi + M \ln \xi], \qquad (1)$$

where

$$\xi = \left(\frac{r}{R}\right)^2, \quad M = \left(1 - \frac{\delta}{R}\right)^2.$$

We also assume that Bernoulli's equation applies to the core flow; in this case in any given section the pressure in the boundary layer is equal to the pressure in the flow core.

From the continuity equation (constant flow rate) we have

$$U_0 = MU + \int_M^1 v\, d\xi. \qquad (2)$$

Substituting the expression in (1) for v and considering that

$$U = v(M, z) = A(z)[1 - M + M \ln M], \qquad (1a)$$

we obtain

$$A(z) = \frac{2U_0}{(M-1)^2}. \qquad (3)$$

Now, applying the change-of-momentum theorem to the mass of liquid flowing in unit time between the inlet section and the section defined by the coordinate z, we have

$$\Delta Q = P - W. \qquad (4)$$

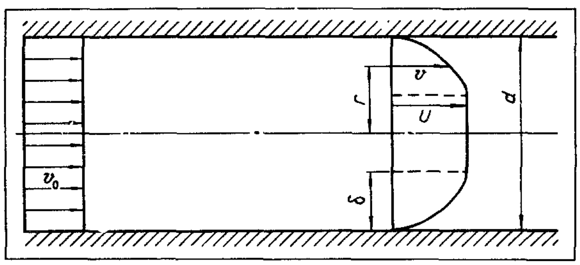

Fig. 1. Velocity profile in entrance region.

Using Bernoulli's equation, we find that

$$P = \pi R^2 (p_0 - p) = \frac{U^2 - U_0^2}{2} \pi R^2 \rho. \qquad (5)$$

With (1) and (3) in mind, we obtain

$$W = 4\pi \int_0^z \mu \left(\frac{\partial v}{\partial \xi}\right)_{\xi=1} dz = 8\pi\mu U_0 \int_0^z \frac{dz}{1-M}. \qquad (6)$$

Finally, we determine the change of momentum from the equation

$$\Delta Q = \pi R^2 \rho \left[MU^2 + \int_M^1 v^2 d\xi - U_0^2\right]. \qquad (7)$$

Substituting v from (1) and U from (1a), after integration we obtain

$$\Delta Q = \pi R^2 \rho \left\{\frac{4U_0^2}{(M-1)^4}\left[\left(-\frac{11}{6}M^2 + \frac{7}{6}M - \frac{1}{3}\right) \times (M-1) + M^2 \ln M\right] - U_0^2\right\}. \qquad (8)$$

Substituting (8), (6), and (5) into (4), after simple transformations we obtain the following equation:

$$\frac{1 + 4M - 11M^2}{3(M-1)^3} + \frac{2(M^3 + M^2 - M)\ln M - M^2 \ln^2 M}{(M-1)^4} - \frac{1}{4} = -\frac{16}{d\,\mathrm{Re}} \int_0^z \frac{dz}{(1-M)}. \qquad (9)$$

Differentiating Eq. (9) with respect to z and then multiplying by $-(1 - M)$, we obtain

$$\left\{\frac{-13 + 20M + 17M^2}{3(M-1)^3} + \frac{(2 + 4M - 12M^2 - 2M)\ln M + 2M(M+1)\ln^2 M}{(M-1)^4}\right\} \times dM = \frac{16\, dz}{d\,\mathrm{Re}}. \qquad (10)$$

Integration of Eq. (10) gives

$$\frac{4z}{d\,\mathrm{Re}} = \frac{19 - 27M}{3(1-M)^2} +$$

Table

Pressure Change and Friction Factor along Length of Entrance Region

$\frac{z}{d\mathrm{Re}}$	$\frac{p_0-p}{\rho U_0^2/2}$	ζ	$\frac{z}{d\mathrm{Re}}$	$\frac{p_0-p}{\rho U_0^2/2}$	ζ
0.0005	0.32	644	0.0065	1.23	186
0.0010	0.46	446	0.0075	1.34	177
0.0015	0.56	373	0.0100	1.57	157
0.0020	0.65	326	0.0150	2.02	134
0.0025	0.72	293	0.0200	2.40	118
0.0035	0.87	248	0.0250	2.73	108
0.0045	1.01	222	0.0295	3.00	100
0.0055	1.12	200			

$$\times \frac{M^2(3+M)\ln^2 M}{3(1-M)^3} + \frac{17M^3-7M^2-2M}{3(M-1)^3}\ln M -$$

$$-\frac{4}{3}\ln M\ln(1-M) - \frac{4}{3}\sum_{k=1}^{\infty}\frac{1}{k^2}M^k + C, \qquad (11)$$

where C is an arbitrary constant.

Since when $z = 0$, $\delta = 0$, and $M = 1$, substituting these values of M and z into (11), we determine the value of C.

Evaluating an indeterminate form of the type 0/0 in accordance with L'Hôpital's rule and noting that $\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$, we find that

$$C \approx -5.862. \qquad (12)$$

Substituting C from (12) into (11) and setting $M = 0$ ($\delta = R$), we find the length of the entrance region:

$$\frac{l}{d\,\mathrm{Re}} \approx 0.0295. \qquad (13)$$

Substituting the values of M, C, and U from (11), (12), and (1a) into Eq. (5), we find the relation between the pressure and the longitudinal coordinate z.

The change of pressure in the entrance region is conveniently expressed in terms of the mean friction factor ζ determined from

$$\frac{p_0-p}{\rho U_0^2/2} = \zeta\frac{z}{d}. \qquad (14)$$

Substituting $(p_0 - p)$ from (5), after simple transformations we obtain

$$\zeta\,\mathrm{Re} = \left[\left(\frac{U}{U_0}\right)^2 - 1\right]\left(\frac{z}{d\,\mathrm{Re}}\right)^{-1}. \qquad (15)$$

When $z \geq l$ Eq. (14) may conveniently be written in the form

$$\frac{p_0-p}{\rho U_0^2/2} = \zeta_0\frac{l}{d} + \xi\frac{z-l}{d}, \qquad (16)$$

where $\xi = 64/\mathrm{Re}$ is the friction factor, constant along the length at $z > l$, and ζ_0 is the friction factor at $z = l$.

Since $\zeta_0\mathrm{Re}$ = const and $l/d\,\mathrm{Re}$ = const, we have

$$\frac{p_0-p}{\rho U_0/2} = \left(\xi\frac{z}{d} + k\right), \qquad (17)$$

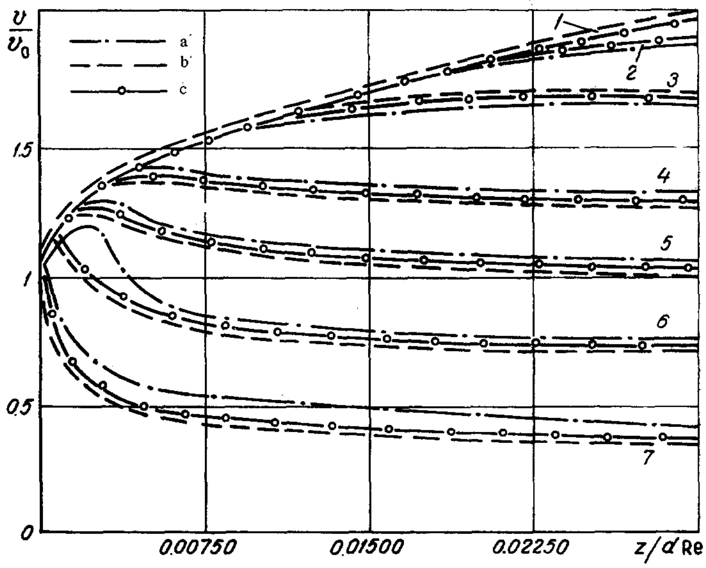

Fig. 2. Velocity as a function of the longitudinal and radial coordinates (a—experiment; b—according to Schiller's calculations; c—from Eq. (1)): 1) r/R = = 0; 2) 0.2; 3) 0.4; 4) 0.6; 5) 0.7; 6) 0.8; 7) 0.9.

where k is a constant. Setting $z = l$, we find that k = = 1.112, which coincides with the recommendations of [4].

According to the data of various authors, k lies in the range from 1.040 to 1.159.

The results of our calculation of the pressure change and the friction factor in the entrance region of the tube are given in the table.

The law of variation of the velocity along the tube axis determined from (1) and (11) is shown in Fig. 2 for various values of (r/R). A comparison with the experimental curves obtained by Nikuradse [1] shows that the proposed method gives good agreement with experiment at values of $z/d \leq 0.0175\,\mathrm{Re}$ for the flow region near the tube axis.

A comparison of this solution with Schiller's solution [2] shows that, apart from the much greater simplicity of the calculations, it is in better agreement with the experimental curves.

The length of the entrance region calculated from (13) is greater than the value obtained by Schiller (0.0287 dRe) and less than the values obtained by Targ [3] (0.04 dRe), Campbell and Slattery [5] (0.0575 dRe), and Boussinesq [6] (0.065 dRe).

A comparison of the solution obtained with the solutions of the above-mentioned authors shows that the friction factor in the entrance region is well described by Eqs. (15) and (17) over its entire length, whereas Schiller's formula gives good agreement with experiment only at $z/d\mathrm{Re} < 0.0075$ and Boussinesq formula only at $z/d\mathrm{Re} > 0.0075$ [6].

The pressure change in the entrance region (table) is in good agreement with [4].

NOTATION

r and z are cylindrical coordinates; ρ is the fluid density; ν is the kinematic viscosity; μ is the dynamic viscosity; δ is the thickness of the boundary layer; d is the tube diameter; Re is the Reynolds number; P and W are pressure and friction forces acting on isolated volume; ΔQ is the change of momentum; P_0 and p denote pressure in inlet section and in investigated section of tube, respectively.

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21 November 1967

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